Last Time: Vector spaces

$V \leftarrow$ set of "vectors"

$+$, $\cdot$
(addition) (scalar mult.)

① $u + v = v + u$

② $u + (v + w) = (u + v) + w$

③ there is a zero-vector 0_v w/ $0_v + v = v$

④ Additive inverses: each v has a $-v$ w/ $v + (-v) = 0_v$

⑤ $(a + b) \cdot v = a \cdot v + b \cdot v$

⑥ $a \cdot (u + v) = a \cdot u + a \cdot v$

⑦ $a \cdot (b \cdot v) = (ab) \cdot v$

⑧ $1 \cdot v = v$

Examples: $\mathbb{R}^n$, $\mathcal{M}_{m,n}(\mathbb{R}) = \left\{ \begin{matrix} m \times n \text{ matrices} \\ \text{w/ entries in } \mathbb{R} \end{matrix} \right\}$,

$\mathcal{P}_n(\mathbb{R}) = \left\{ \begin{matrix} \text{degree} \leq n \text{ polynomials} \\ \text{w/ coefficients in } \mathbb{R} \end{matrix} \right\}$, + sporadic examples.

$\text{Func}(S, \mathbb{R}) = \{ \text{functions } S \to \mathbb{R} \}$ ← Important example!

Prop: Let V be a vector space w/ $v \in V$ and $c \in \mathbb{R}$.

① $0 \cdot v = 0_v$ ② $-v = (-1) \cdot v$ ③ $c \cdot 0_v = 0_v$.

pf: Let V be a v.s. w/ $v \in V$ and $c \in \mathbb{R}$.

① $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v$ so subtracting $0 \cdot v$ from both sides yields $0_v = 0 \cdot v$.

② $0_V = 0 \cdot v = (1 + (-1)) \cdot v = 1 \cdot v + (-1) \cdot v$

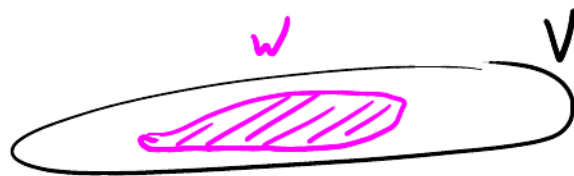
So $0_V = v + (-1)\cdot v$ and subtracting v from both sides yields $-v = (-1)\cdot v$.

③ $c \cdot 0_V = c \cdot (0_V + 0_V) = c \cdot 0_V + c \cdot 0_V$, so subtracting $c \cdot 0_V$ from both sides yields $0_V = c \cdot 0_V$ ▨

Subspaces

Idea: Find vector spaces within our vector spaces!

Def: Let V be a vector space. A subspace of V is a subset $W \subseteq V$ which is itself a vector space under the operations on V, restricted to W.

Unpacking This Definition:

this "restricted operations" thing:

$+ : V \times V \to V : (u,v) \mapsto u+v$

$\cup\!\!\mid \quad \cup\!\!\mid \quad \cup\!\!\mid$

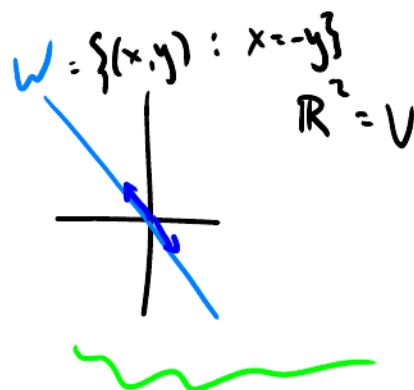
$+ : W \times W \to W$

point: want addition of vectors in W to stay in W.

We also need scalar mult of vects in W to "stay in" W...

$\cdot : \mathbb{R} \times V \to V : (r,v) \mapsto r \cdot v$

$\cup\!\!\mid \quad \cup\!\!\mid$

$\cdot : \mathbb{R} \times W \to W$

Ex: Let $V = \mathbb{R}^3$ and $P = \{(x,y,z) \in \mathbb{R}^3 : x - y + 3z = 0\}$

Then P is a subspace of $\mathbb{R}^3$. To see this, we need to verify that P is a v.s. under the restricted operations from $\mathbb{R}^3$...

Almost missed closure!

① ✓ (Comm): + is comm on $\mathbb{R}^3$, it remains so in rest.

② ✓ (Assoc,+): + is assoc on $\mathbb{R}^3$, so too on P.

③ (zero): We need to show $0_{\mathbb{R}^3} \in P$. Indeed:

need! $(x,y,z) = (0,0,0)$ satisfies $0 = 0 - 0 + 3 \cdot 0 = x - y + 3z$.

Hence the zero-vector $(0,0,0) = 0_{\mathbb{R}^3} \in P$.

Closure: Suppose $(x_1, x_2, x_3), (y_1, y_2, y_3) \in P$ and $c \in \mathbb{R}$.

Need: $(x_1, x_2, x_3) + (y_1, y_2, y_3) \in P$ $\quad P + P \subseteq P$

and $c \cdot (x_1, x_2, x_3) \in P$

Addition: $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$ needs to satisfy

$$(x_1 + y_1) - (x_2 + y_2) + 3(x_3 + y_3) \stackrel{?}{=} 0.$$

Now $(x_1 + y_1) - (x_2 + y_2) + 3(x_3 + y_3)$

$= (x_1 - x_2 + 3x_3) + (y_1 - y_2 + 3y_3)$ $\quad x - y + 3z = 0$

$= 0 + 0 = 0$ as desired.

Scalar Multiples: $c \cdot (x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$ satisfies (Need)

$cx_1 - cx_2 + 3cx_3 = c(x_1 - x_2 + 3x_3) = c \cdot 0 = 0$,

so $c \cdot (x_1, x_2, x_3) \in P$ as desired.

Point: P is closed under + and $\cdot$.

④ (Negatives): $(-1)\cdot v = -v$, so closure under scalar mult yields negatives as desired...

⑤ ("Left dist"): $a\cdot(u+v) = a\cdot u + a\cdot v$ in $\mathbb{R}^3$, so it's true in P.

⑥ ("Right dist"): $(a+b)\cdot v = a\cdot v + b\cdot v$ in $\mathbb{R}^3$, so it holds in P

⑦ ("assoc" for $\cdot$): $a\cdot(b\cdot v) = (ab)\cdot v$ in $\mathbb{R}^3$, so again in P!

⑧ ("Identity"): $1\cdot v = v$ so holds automatically in P.

V

P

$P \subseteq V$

Prop (Subspace Test): Let V be a vector space and let $S \subseteq V$. The following are equivalent.

① S is a subspace of V.

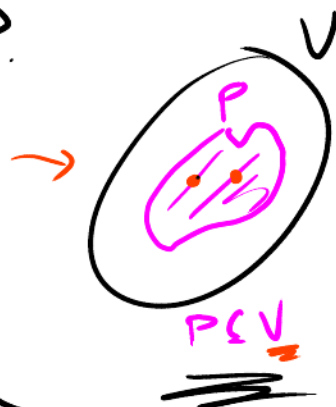
② S is closed under addition and scalar multiplication and $0_V \in S$.

NB: The proof was (in spirit) already done when we discussed $P \subseteq \mathbb{R}^3$ above.

Point of Subspace Test: If we want to show $S \subseteq V$ is a subspace of V, we only need to check three things: ① $0_V \in S$, ② S is closed under addition, ③ S is closed under scalar multiplication.

Ex: The trivial subspace of any vector space V is $\{0_V\} \subseteq V$. Let $S = \{0_V\}$. We know

① $0_V \in S$ ② $0_V + 0_V = 0_V$ so S closed under $+$

③ $c\cdot 0_V = 0_V$ so S is closed under scalar mult! ☑

Ex: Let $S = \{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w=0\}$.

Let's use the subspace test to show S is a subspace of $\mathbb{R}^4$.

① $0+0+0+0=0$ so $0_{\mathbb{R}^4} = (0,0,0,0) \in S$.

② Let $(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) \in S$.

Then $x_1+y_1+z_1+w_1 = 0 = x_2+y_2+z_2+w_2$.

Hence $(x_1+x_2)+(y_1+y_2)+(z_1+z_2)+(w_1+w_2)$

$= (x_1+y_1+z_1+w_1)+(x_2+y_2+z_2+w_2)$

$= 0+0 = 0$

Thus $(x_1,y_1,z_1,w_1)+(x_2,y_2,z_2,w_2) \in S$, and we see S is closed under vector addition!

③ Let $(x,y,z,w) \in S$ and $c \in \mathbb{R}$. Now

$x+y+z+w=0$, so

$$cx+cy+cz+cw = c(x+y+z+w) = c \cdot 0 = 0$$

Hence $c\cdot(x,y,z,w) \in S$ and S is closed under scalar multiplication!

Hence S is a subspace of $\mathbb{R}^4$ by the subspace test! ▯

<u>Notation</u>: We write "$S \leq V$" to mean "S is a subspace of V". That symbol is **NOT** the same as $S \subseteq V$ (← subset) because these

aren't the same concept!

Non-Ex: Let $S = \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \in \mathbb{R}^2 : x \in \mathbb{R} \right\}$.

$S \subseteq \mathbb{R}^2$ is a subset of $\mathbb{R}^2$.

But $S \not\leq \mathbb{R}^2$ (i.e. S is not a subspace of $\mathbb{R}^2$)

because... ① $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ x \end{pmatrix}$ for any x...

② $\begin{pmatrix} 1 \\ x \end{pmatrix} + \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ x+y \end{pmatrix} \neq \begin{pmatrix} 1 \\ z \end{pmatrix}$ for any z.

③ $c \cdot \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} c \\ cx \end{pmatrix} \in S$ if $c = 1$.

So S fails all three conditions... ☒

Ex: The trivial subspace of any vector space V is $\{0_V\} \subseteq V$. Let $S = \{0_V\}$. We know

① $0_V \in S$ ② $0_V + 0_V = 0_V$ [so S closed under $+$]

③ $c \cdot 0_V = 0_V$ so [S is closed under scalar mult!] ☒

$\subseteq \mathbb{R}^2$ bad, b/c not closed under $+$.

V, 0_V, W

S

$\frac{1}{2}v$, v

$S \subseteq \mathbb{R}^2$

$0_V \in S$

$u, v \in S \Rightarrow u + v \in S$

S NOT closed under scaling.